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THE DIDACTICS OF MATHEMATICS: ITS PROBLEMATIC AND RELATED RESEARCH

My ultimate goal today is to give a general idea of the style of research that is being done by the Marseille group. But before I come to this point I must make clear the framework of our research effort. In order to give insight into it I shall have to draw upon a central concept of epistemology, that of “problematic”. In French, the word “problematic” is normally used as an adjective, as I suppose it is in English. As far as I know, it was the philosopher and epistemologist Gaston Bachelard who first used it (in French) as a noun, in order to express the unifying perspective common to what he had until then simply termed a “body of problems”.

So, in what follows my main aim will be to describe what I shall call the *general problematic of the didactics of mathematics*. In my opinion the specificity of the didactics of mathematics, as some of us understand it, should be pointed out by contrasting it with that of pedagogy. In France pedagogy has been made the object of some explicit theoretical analyses, of which the most famous supporter was the sociologist Emile Durkheim. Writing on the theme of the nature of pedagogy, he stressed that school-reformers always act under the pressure of necessity and therefore have no time to waste in theoretical or scientific thinking about teaching and instruction. According to him, in such an emergency (which had arisen, if we follow his reasoning, by the end of the 19th century, and still exists, to follow present-day school-reformers, in the closing decades of this century), no time can be allotted to the search for an answer to the question “How does it work?”. The available energy and capacities should best be used to answer the straightforward and simpler question “How should it work?”. The definition of educational objectives, along with the ways and means to achieve the assigned goals, are the main constituents of “pedagogy”, of what it is or at least should be. It could be shown that, although shaped by a different concern and style of thought (less sociological, more psychological), Piaget’s opinion on what pedagogy is about differs very little from Durkheim’s: pedagogy is (in Durkheim’s own terms) a “practical theory”, providing the teacher with rules and criteria concerning suitable ways to set up the most effective teaching conditions. Given this purely pragmatic approach, pedagogy doesn’t aim at developing a true science on its own, it is simply thought of as a set of workable “truths”, to be obtained through the *application* of existing sciences (such as sociology, psychology and so on) to specific problems of education.

In this line of thought, the description of what pedagogy at large is about can assume different aspects, but it is always the same character in a different guise. The idea of a theory of education with purely pragmatic ends, working its way through the tremendous complexity of classroom situations thanks to the crutches provided by psychology, sociology, and the like, is a deep-rooted one. It has been supported by the finest minds. In this country, I have found a quite clear statement of the doctrine by one of the most famous researchers in the field of education. Professor Bruner of Harvard University, in a paper written some years ago, gives the following definition of the theory he is aiming at:

“A theory of instruction is prescriptive in the sense that it sets forth rules concerning the most effective way of achieving knowledge or skill. By the same token, it provides a yardstick for criticizing or evaluating any particular way of teaching or learning.

A theory of instruction is a normative theory. It sets up criteria and states the conditions for meeting them. The criteria must have a high degree of generality: for example, a theory of

*instruction should not specify in ad hoc fashion the conditions for efficient learning of third-grade arithmetic; such conditions should be derivable from a more general view of mathematics learning.”*¹

Indeed, such a “practical theory” should be the ultimate goal of all those working in the field of education. But history unfortunately shows that all efforts to come to grips in a direct way with the pragmatic problems of education have until now produced only aborted “theories”. The ambition to build up a set of rules and criteria allowing successful decision-making in the classroom is definitely a premature one. What is needed, as history teaches us, is a theoretical “detour”. It is fairly easy to grasp the idea, if we consider the case of the physical sciences as an illustration of this argument: long before Galileo was able to state the first correct laws of mechanics, military engineers had devised and constructed mechanical weapons. They could build them, and use them, but they could not explain why they did, or did not, work. Only many centuries later was a scientific explanation made available, thanks to “modern physics”. In between, there was the awakening of physics, what some historians call a “scientific revolution”. Erudite treatises had been written on machines, but almost no scientific knowledge had been developed before Galileo. Leonardo was indeed an engineer – he built all types of machines, weapons and other mechanical devices. Galileo was a scientist, in the modern sense of the word. Engineers had to meet practical and often pressing needs: in a besieged city they had to invent in a very practical perspective... No time was left for theoretical speculation. But it must be emphasized that in the long run, in order to go from the ancient catapult, already known to the Greeks, to modern weapons, a theoretical “detour” was needed. It is this necessity we cannot pretend to ignore. The teacher can be compared to a military engineer, and as such engaged in a continuing activity to devise didactic conditions adapted to classroom processes. But whatever his skill, his powers remain limited.

Given all this, my contention is that we cannot escape the vicious circle of immediate action, and consequently limited means of action, unless we break the spell of immediate practical success in the classroom and get rid of the short-sighted logic of success and failure, in an attempt to take a scientific view of the task ahead of us. Of course, this will not be done with good intentions only... In what follows I shall try to describe some criteria to be imposed upon any scientific approach to educational matters. Let me stress however how this tentative venture may prove difficult, considering it in the light of other similar historical situations of dawning sciences. Piaget has rightly remarked that work in any new scientific field

*“is work requiring patience and careful research which can only slowly vanquish our mental habits oriented toward speculation of the whole. In this respect, the great danger is to construct too rapidly and after the first efforts to give in to the fascination of the spirit of system”*².

Unfortunately Piaget’s concern here is about his own project of a scientific epistemology. I mentioned previously Piaget’s view as regards education: it is much less clear. The major difficulty we shall have to face is the unavoidable gap between what society expects and what science can give. To his contemporaries Galileo, who devoted so much time and effort to thinking and theorizing about futile problems, such as dropping and tossing objects, appeared to be something of a madman. But as we know today, he was right. His opponents, who tried to ridicule him, were definitely wrong. The discrepancy between the two priorities, one pragmatic and aiming at immediate practical success, the other scientific and aiming at

¹ *Toward a Theory of Instruction*, Harvard University Press, Cambridge (Mass.), 1978, p. 40.

² *Psychology and Epistemology*, The Viking Press, New York, 1971, p. 99.

theoretical explanation, often led supporters of a new scientific domain to endorse casual quackery. As an example I suggest we look at the state of medicine by the end of the 17th century: little was known, however physicians had to pretend to cure cancers, tuberculosis, etc., in an effort to secure financial support and social legitimacy. The trickery was inescapable in order to know a little more... A less remote example is offered by meteorology. In France not long ago, meteorology was supposed to be able to forecast the weather for weeks to come, and indeed most weather forecasts (issued by radio stations) took into account this popular requirement. For a number of months now, there has been an attempt to accustom audiences to the fact that nothing can be said with a reasonable degree of certainty beyond the next couple of days. This has perhaps been made possible by the increasing sophistication of meteorology, and its consequently increased credibility among the general public. Any new science must by necessity indulge in some make-believe, until it is strong enough to meet the demands of society, or at least until it can publicly make clear what can, and what cannot, be demanded at the given stage of development.

In order to avoid any misunderstanding I shall emphasize one more general point. One must not contrast the didactics of mathematics with pedagogy by describing it as indifferent to practical success and teaching practice. It duly intends to set forth rules for the teacher and to cope with the practical difficulties of the teaching process. But the teacher and his teaching practice must receive a scientific status in the theoretical analysis didactics is seeking to provide, before we can translate our knowledge of “how it works” into the language of action, i.e. before we can turn didactic knowledge into practical teaching advice. In fact, the viewpoint expounded so far can economically be summed up by the postulate that any fruitful scientific study of some order of reality operates a dialectic (i.e. a productive interaction) between “practice” and “theory”. In other words it brings into being a synthesis between the realm of thought and the realm of action. This old view of the knowledge-process has been put in more concrete terms by the french epistemologist G. Bachelard. In an attempt to analyse the essential features of modern science, Bachelard points to the fact that science no longer complies with the clear-cut program of some simple scientific *phenomenology* according to which science would merely aim at a *description* of phenomena. Instead complete scientific knowledge can not be ascertained until we have gained mastery over the phenomena themselves by *producing* them. Therefore science is not simply descriptive, it is also *productive*, and in this sense only it can be said to provide complete explanation. To distinguish the productive aspect of science from its commonplace descriptive power, Bachelard has coined the word *phenomenotechnics*. The didactics of mathematics really aim at being a phenomenotechnics in this sense. It intends to master the didactic process, both theoretically and practically, because practical mastery cannot be secured unless theoretical control is achieved.

A word of caution would not be out of order here. It should *not* be understood from the preceding considerations that in the double-faced mastery just described, theoretical speculation takes precedence over practical ends, in agreement with the widespread empiricist conception of “pure science” followed by “applications”. It is in the very building of scientific knowledge that these two aspects are closely intermingled. In this context a policy of separation of theory and practice would prove to be hardly workable.

The shift in scientific perspective which we are trying to bring about can be made more concrete if we turn to the language and concepts of *system theory*. The scientific study of didactic processes brings to the fore what may be called a *didactic system*, which is to be the main concern of the didactician. What exactly this “didactic system” is made of is the chief

issue in this respect. To be sure, answers to this question will be a highly revealing criterion according to which a rough but illuminating hierarchy can be sketched. The crudest form is one in which the didactic system (as a theoretical model) is seen as made up of just one “part”, which is *knowledge*. This knowledge-centered perspective is basically the teacher’s perspective, although it usually remains implicit. In this model, it can easily be shown that the student as a psychological subject receives limited consideration, which must be understood in a theoretical sense, with no moral or humanist implications whatever. The next step is taken when one becomes aware that the teaching process is concerned in the first place with the child, or the student. This step has been taken in a deep emotional context when, in the first decades of this century, the child suddenly appeared as the great discovery of the period. The names of the Swedish feminist Ellen Key, whose book (1909) heralded *The Century of the Child*, and for this country, of John Dewey, whose book *My Pedagogic Creed* was a major contribution to educational philosophy, duly epitomize the attitudes then prevailing toward education. This change was further shaped by the appeal to experimental and genetic psychology. In the beginning much was expected, but it seems that in the end little, or too little, was obtained in this way. Let me remind you that such a distinguished expert as Professor Thorndike of Columbia University in the years before World War II still advocated rote teaching as a scientifically supported teaching strategy. However, one fact worthy of note is that the interest taken in the study of learning processes has soon induced changes in psychological research itself by drawing the psychologist’s attention to the student’s *procedures*, as different from the general cognitive *structures* so far studied. This last fact should carefully be related to what I previously called the specificity of the didactics of mathematics: the change in the object of study brings about changes in the problems investigated.

A genuine interest in the subject’s procedures and thought operations brings to the fore something which tended to be forgotten in the perspective first described. Essentially it points out the active role played by the student in the construction of his own knowledge. Earlier doctrines, either philosophical or pedagogical, made implicit or explicit assumptions about the “normal” subject’s mental faculties. But as Piaget duly remarks

*“these doctrines, although careful to characterize the qualities they attributed to this instrument (the reminiscence of ideas, the universal power of Reason, or the character both previous and necessary of the a priori forms), neglected to verify that it is was actually at the subject’s disposal (...) It is evident that, before conferring such ‘faculties’ on ‘every’ normal human being, it would be worthwhile to examine them.”*³

However, although this type of investigation is necessary and actually fruitful, it still leaves us half way from our true goal, which is the construction of a theory of didactic processes, giving us practical mastery over classroom phenomena. At this point we can recall the distinction between pedagogy and the didactics of mathematics: it is my opinion that the dividing line between them separates those conceptions in which the teacher remains outside the system under consideration, and those in which the teacher is included as a new theoretical “part” in the didactic system, together with knowledge and student. Again, one must be cautious not to attach humanist implications to the fact that the teacher thus becomes an abstract function in an abstract model purporting to make reality understandable to us. It is my contention that this move alone can help to lay the foundations of the didactics of mathematics as a new scientific field distinct from previous straightforward prescriptive pedagogical doctrines.

³ Piaget, *op. cit.*, p. 5.

At this point the didactic system is made up of three parts: knowledge, student and teacher. Three models evolve from the description given so far. In model I, all phenomena (including the subject's behaviour) are interpreted in terms of knowledge alone; I shall ascribe it to the teacher and call it the teacher's model. In model II the subject is introduced, and the active role is taken over by the student, while knowledge tends to be taken for granted. I shall credit model II to the psychologist and call it the psychologist's model. Finally, model III takes into account at the same time the knowledge, the student and the teacher, and should be assigned to the didactician, on condition that one more characteristic be included in its definition, that of being an *open system* (in the sense of system theory). Referring to physical chemistry, von Bertalanffy terms a system "closed" if no material enters or leaves it; it is called "open" if there is import and export of material⁴. It is noteworthy that this definition still applies to didactic systems if we consider the exchanges which take place between them and their environment (which comprises society at large as well as specialised subgroups, such as experts, parents, all possible commissions and committees, and so on): a didactic system maintains itself in an almost continuous inflow and outflow of that particular material, knowledge. In fact, for reasons pertaining both to the proper functioning of the system and to the pressure exerted on it by the surrounding milieu, the knowledge to be imparted within the system gradually becomes obsolete and is eventually discarded; while new knowledge, or old knowledge in new form, enters curricula.

The classification just delineated appears as a hierarchy of models in so far as the internal complexity of models is used as a classifying principle. It should not be concluded too rapidly that in going from model I to model III one passes from simple-mindedness to intellectual sophistication. On the one hand, each of the models may prove useful in given practical situations: for example what I have been calling the teacher's model will be rightly drawn upon by the teacher, whenever other possibly relevant didactic variables (such as which procedures are being used by students) remain beyond his control. Indeed, it is a problem in the didactics of mathematics to explain to what extent and under what conditions recourse to model I actually helps the teacher come to an optimal decision in a given didactic situation. On the one hand, what makes a system-approach necessary, i.e. the existence of interactions between the different parts of the system, may be weak enough to be neglected for certain research purposes. Under this condition the classical analytical procedure still applies.

However, it should be stressed that in investigating most didactic situations, a model III-approach proves necessary. Interactions between parts cannot be ignored. This is not to say that *every element* in the system should be taken into account (or accounted for). In fact, the approach so far advocated can rightly be termed an anti-empiricist approach, in which relevancy comes defined as a result of concrete analysis. Thus, if any didactic system is to be considered as an open system in the sense defined above, didactic analysis does not, for all that, encroach on the sociology of education: *some* sociological elements may prove relevant in some cases, but appeal to sociological (or historical) explanation is not a must and should in every case be justified. This idea amounts to the methodological principle according to which a didactic fact can only be explained by another didactic fact (a principle borrowed with a slight change from sociologist E. Durkheim who held that social facts should be explained by social facts). Of course the exact meaning of this principle depends on what a "didactic fact" exactly is. On the one hand, not all "facts" which happen in the classroom (more specifically, in the mathematics classroom) are didactic facts. A didactic fact is relative to a specific concept or subject matter, so that a change in the concept involved produces a change in the "fact" itself. For instance, a change of temperature in the classroom, which

⁴ Cf. Ludwig von Bertalanffy, *General System Theory*, Penguin Books, 1973, p. 128.

certainly does alter the general conditions of activity, will not be regarded as a *didactic* fact (unless it is duly proved to be so) because we have every reason to believe that it alters the physical and mental conditions irrespective of the mathematical concepts involved. In other words, temperature is a general factor which may influence the pupil's activity, but it is not a *didactic variable* in the sense just defined, i.e. it certainly does not result in a change of the didactic *structure*. On the other hand, as emphasized previously, "didactic facts" may involve much more than just what happen in the classroom, due to the constraints imposed upon the inner functioning of the didactic system by the milieu at large.

With this in mind, I proceed to give a very rough sketch of the work done by the Marseille group. I shall retain here just one point, to serve as an illustration of the research problems, made apparent by the problematic described so far. It is concerned with that "part" of the didactic system which I have simply termed "knowledge". As often as not, the status of knowledge in the didactic process is taken for granted on the ground that whatever will be taught (e.g. the number concept, quadratic equations, algebraic identities, etc.) has been defined beforehand (by mathematicians) regardless of its being taught. In accepting this straightforward view, the didactician (who stands normally outside the system in order to analyse it) would take over what is a necessary myth shared by those standing inside the system (e.g. the teacher), i.e. the myth of the identity of what is actually taught with what is supposed to be taught. Indeed, the teaching of a given mathematical subject induces changes in the subject itself in order to make that mathematical subject teachable, that is compatible with the structure of the didactic system and the didactic processes. Sometimes the changes simply amount to the creation of new teaching items: thus the "New Math" among other things has created, or at least has raised from obscurity, *Venn diagramms*, a long-forgotten trick unknown to present-day mathematicians (on Venn diagramms I refer to Freudenthal's acute criticism). Usually, however, knowledge is processed in a much less conspicuous way, but the processing still exists. The reason for that *didactic processing*, also called the didactic *transposition* of knowledge, lies in the necessity of some sort of compatibility between the different parts of the didactic system. Let me mention here only one of the constraints to be satisfied. Every "teaching object" i.e. every object of knowledge such as it is after the didactic processing has taken place, must be compatible with the *general structure of didactic time*. This actually means that when it is introduced by the teacher at some point in time, each teaching "object" should first appear as something *new*, so that a (generally implicit) agreement can be reached between teacher and students as concerns the acceptability of this object of knowledge as a "teaching object", which means they do come (in fact, tacitly) to the agreement that something is to be taught (by the teacher) and something is to be learnt (by the students). Secondly, the proposed item must at the same time seem *old*, including features lending themselves to recognition, so that students feel not altogether unfamiliar with it. Failure to meet these conditions would result in a breach of the *didactic contract* that normally brings together teacher and students around a common teaching and learning project. This very general condition may still be expressed by saying that a teaching item has to be or, more precisely, must appear as, a "transactional" object, which fills the gap between past and future, thus allowing the student to go from what he knows (or, more accurately, what he is already familiar with) to new knowledge with a sense of continuity. Every teaching item when viewed as located at some point in the didactic duration, assumes a contradictory nature, and this general functional constraint entails and, at the same time, explains, many peculiarities of the didactic processing of knowledge. Of course, for every object of knowledge retained as an "object to be taught" it will be necessary to carry out a detailed analysis specific to the knowledge involved. Although this type of analysis may be of interest in its own right, I wish to emphasize here that in some cases it leads to the proper

understanding of the pupil's failures, in a way that remains free from any touch of psychologism. But this, allow me to remind you, will only be achieved through a series of carefully detailed monographic studies, which we have already accomplished in a number of cases with – in my opinion! – quite satisfactory results.

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